SHARP WEIGHTED ESTIMATES FOR MULTI-FREQUENCY CALDERÓN-ZYGMUND OPERATORS

SAURABH SHRIVASTAVA AND SENTHIL RAANI K. S.

Abstract. In this paper we study weighted estimates for the multi-frequency $\omega-$Calderón-Zygmund operators T associated with the frequency set $\Theta = \{\xi_1, \xi_2, \dots, \xi_N\}$ and modulus of continuity ω satisfying the usual Dini condition. We use the modern method of domination by sparse operators and obtain sharp bounds $\|T\|_{L^p(w)\to L^p(w)} \lesssim N^{\frac{1}{2}}[w]_{\mathbb{A}_p}^{max(1,\frac{1}{p-1})}$ for the exponents of N and $\mathbb{A}_p$ characteristic $[w]_{\mathbb{A}_p}$.

1. Introduction

The famous $\mathbb{A}_2$ conjecture about the linear growth of the $\mathbb{A}_2$ characteristic constant $[w]_{\mathbb{A}_2}$ in weighted norm inequalities for Calderón-Zygmund operators has been resolved recently in [9] by T. Hytonen. It took considerable amount of time and efforts of a rather large group of mathematicians in order to have a better understanding of the problem. It would be a tedious task to even state all the recent works in the theory. We would only highlight the most relevant works in the context of the current paper. We first try to recall some of the important developments in this direction prior to the seminal paper of T. Hynonen. In [18] and [19] S. Petermichl proved the conjecture for the Hilbert transform and the Riesz transform respectively using representation of the Hilbert transform in terms of the Haar shift operators. Around the same time O.V. Beznosova obtained linear bounds for $\mathbb{A}_2$ characteristic constant in weighted inequalities for the dyadic paraproduct operators. Later, in [12], [6] and [7] authors proved the conjecture for the Haar shift operators. Finally, in [9] T. Hytonen settled the $\mathbb{A}_2$ conjecture for Calderón-Zygmund operators in its general form.

The modern technique in the theory of Calderón-Zygmund operators involves domination of the Calderón-Zygmund operators by sparse operators. This technique yields sharp constants in weighted norm inequalities for these operators with respect to the $\mathbb{A}_p$ characteristic. The technique of domination by sparse operators has evolved in its current form after constant efforts from a large group of mathematicians. The domination of the the Calderón-Zygmund operators by sparse operators was first introduced by A. Lerner [13], where the author obtained a new proof of the $\mathbb{A}_2$ conjecture for the $\omega-$Calderón-Zygmund operators with ω satisfying the logarithmic Dini condition

$$\int_0^1 \omega(t)\log\frac{1}{t}\frac{dt}{t} < \infty.$$

Date: October 4, 2016.

2010 *Mathematics Subject Classification.* Primary: 42B20; Secondary: 42B25.

Key words and phrases. Calderón-Zygmund operators, sparse operators, $\mathbb{A}_p$ weights.

He showed that such operators can be dominated by sparse operators in norm of an arbitrary Banach function space over $\mathbb{R}^n$. The $\mathbb{A}_2$ conjecture in the context of sparse operators was known due to D. Cruz-Uribe, J.M. Martell, and C. Perez [6]. Hence, the domination of Calderón-Zygmund operators by sparse operators in [13] immediately yields the sharp weighted bounds. Soon after the paper of A.K.Lerner [13] the methods of domination by sparse operators were strengthen further by J. Conde-Alonso and G. Rey [5] and A. Lerner and F. Nazarov [15]. They obtained pointwise domination for the $\omega-$Calderón-Zygmund operators with ω satisfying the logarithmic Dini condition. Finally, in [11] M. Lacey extended their result for operators associated with ω satisfying the usual Dini-condition

$$\|\omega\|_{Dini} := \int_0^1 \omega(t)\frac{dt}{t} < \infty. \tag{1.0.1}$$

In [10], authors gave yet another proof of $\mathbb{A}_2$ conjecture while capturing the dependence of the operator norm $\|T\|_{L^2(w)}$ on Dini constant $\|\omega\|_{Dini}$. This result was further simplified by A.K. Lerner [14]. In this paper we closely follow the arguments given in [14] in order to prove Theorem 2.1-the main result of this paper.

1.1. **Multi-frequency Calderón-Zygmund operators.** In the study of Calderón-Zygmund operators the use of the local oscillation $f - \frac{1}{|Q|}\int_Q f$ for Q a cube (or ball) in $\mathbb{R}^n$ explains the role of frequency 0. The situation, when one has to consider the oscillation for functions modulated by frequencies coming from a finite set of $\mathbb{R}^n$, naturally arises in time-frequency analysis and have to be dealt with simultaneously. F. Nazarov, R. Oberlin and C. Thiele [16] introduced an appropriate Calderón-Zygmund decomposition adapted to N frequencies. This decomposition is an extremely useful tool to study the case of multiple frequencies. We shall provide the precise statement of the decomposition shortly. In [2], F. Bernicot extended boundedness of the Calderón-Zygmund operators relatively to a collection of multiple frequencies, focusing on sharp constants with respect to the number of considered frequencies.

Let $\Theta := \{\xi_1, \xi_2, ...\xi_N\}$ be a collection of N frequencies of $\mathbb{R}^n$. An L^2 bounded linear operator $T = \sum_{j=1}^N T_j$, with norm $\|T\|_2$ independent of N, is said to be a multi-frequency Calderón-Zygmund operator relatively to the collection Θ if there exists kernels $(K_j)_{j=1,2...N}$ satisfying the regularity condition: for every $x \neq y$,

$$\sum_{j=1}^N \left|\nabla_{(x,y)} e^{i\xi_j\cdot(x-y)} K_j(x,y)\right| \lesssim |x-y|^{-n-1}, \tag{1.1.2}$$

where the integral representation of T_j is given by

$$T_j(f)(x) = \int_{\mathbb{R}^n} K_j(x,y) f(y) dy,$$

for every function $f \in L^2(\mathbb{R}^n)$ with compact support and $x \in \operatorname{supp}(f)^c$.

Example 1.1. *A typical example of multi-frequency Calderón-Zygmund operators arises from a covering of the frequency space. Let $(Q_j)_{j=1,2...N}$ be a family of disjoint cubes and ϕ_j be a smooth function with $\hat{\phi}_j$ supported and adapted to Q_j.*

Then the linear operator given by

$$T(f) = \sum_{j=1}^{N} \phi_j * f$$

is a multi-frequency Calderón-Zygmund operator associated with the collection $\Theta := \{\xi_1, \dots, \xi_N\}$ *where for every* j, $\xi_j = c(Q_j)$, *the center of the cube* Q_j. *We refer the interested reader to* [2] *for more details.*

Note that the operator $T = \sum\limits_{j=1}^{N} T_j$, being sum of N operators, trivially satisfies L^p-estimates with the operator norm bounded by a constant of order N. However, obtaining sharp constants for the parameter N requires the multi-frequency Calderón-Zygmund decomposition as introduced by F. Nazarov, R. Oberlin and C. Thiele in [16].

Theorem 1.2. [16] *(Calderón-Zygmund decomposition for multiple frequencies): Let* $\Theta = \{\xi_1, ..., \xi_N\}$ *be an arbitrary collection of* N *frequencies of* $\mathbb{R}^n$ *for some* $N \geq 1$. *For* $f \in L^1(\mathbb{R}^n)$ *and* $\lambda > 0$, *there is a universal constant* C *such that the following decomposition holds*

$$f = g + \sum_{Q \in \Omega} b_Q,$$

where $\Omega = (Q)$ *is a collection of disjoint cubes with the following properties.*

- *the collection* $(3Q)_{Q\in\Omega}$, *where* $3Q$ *denotes the cube with the same center as* Q *and three times the side-length of* Q, *has a bounded overlap;*
- *for each* $Q \in \Omega$, b_Q *is supported in* $3Q$*;*
- *we have*
$$\sum_{Q\in\Omega} |Q| \leq CN^{1/2} \|f\|_{L^1} \lambda^{-1};$$
- *the "good part"* g *satisfies*
$$\|g\|_{L^2}^2 \leq C \|f\|_{L^1} \lambda N^{1/2};$$
- *the cubes* Q *satisfy*
$$\|f\|_{L^1(Q)} \leq C|Q|\lambda N^{-1/2} \text{ and } \|f - b_Q\|_{L^2(Q)} \leq C\lambda |Q|^{1/2};$$
- *we have cancellation for all frequencies of* Θ*: for all* $j = 1, 2...N$ *and* $Q \in \Omega$, $\hat{b}_Q(\xi_j) = 0$.

Moreover, the exponent of N *in the above is optimal.*

The proof of the multi-frequency Calderón-Zygmund decomposition relies on a beautiful result of P. Borwein and T. Erdélyi [4] about the reverse Hölder-type inequality for the shift invariant functions in L^p. This decomposition played a key role in the paper [16] where the authors have studied variational norm estimates for exponential sums. The authors had anticipated a wide range of applications of their decomposition in time-frequency analysis. Indeed, it turns out to be an effective and useful tool in order to deal with operators which have more than one preferred frequency or point of singularity. For example, in [17], authors used a discrete variant of the multi-frequency Calderón-Zygmund decomposition in order to obtain uniform estimates for the Walsh model of the bilinear Hilbert transform.

In [2], F. Bernicot exploited the decomposition given in Theorem 1.2 and obtained the following sharp L^p estimates for the multi-frequency Calderón-Zygmund operators.

Theorem 1.3. [2] *Let $\Theta = \{\xi_1, \xi_2, ...\xi_N\}$ be a collection of N frequencies of $\mathbb{R}^n$ and T be the multi-frequency Calderón-Zygmund operator relatively to Θ. Then*

- *T is of weak type $(1,1)$ with $\|T\|_{L^1\to L^{1,\infty}} \lesssim N^{1/2}$.*
- *for $p \in (1,\infty)$, T is bounded on L^p with $\|T\|_{L^p\to L^p} \lesssim N^{|\frac{1}{p}-\frac{1}{2}|}$.*

We point out in the next section (Theorem 2.2) that the above theorem can be extended to the case of multi-frequency operators whose kernels satisfy a weaker regularity condition, namely the Dini-type regularity condition (1.0.1) instead of the regularity condition (1.1.2). Let $\omega : [0,1] \to [0,\infty)$ be a modulus of continuity, that is, ω is increasing, subadditive, $\omega(0) = 0$ and satisfies the Dini condition (1.0.1). We shall refer to a multi-frequency Calderón-Zygmund operator relatively to a frequency set Θ as a multi-frequency $\omega-$Calderón-Zygmund operator relatively to the set Θ if it satisfies the following Dini regularity condition.

$$\sum_{j=1}^{N}(|K_j(x,y) - K_j(x',y)| + |K_j(y,x) - K_j(y,x')|) \leq \omega\Big(\frac{|x-x'|}{|x-y|}\Big)\frac{1}{|x-y|^n} \tag{1.1.3}$$

for all x, x' and y such that $|x-y| > 2|x-x'|$.

F. Bernicot [2] had proposed some new ideas to obtain weighted estimates for the generalized Bochner-Riesz means by using the multi-frequency Calderón-Zygmund operators. He obtained some partial results for weighted boundedness of the multi-frequency Calderón-Zygmund operators. In order to describe these results, we recall the notion of $\mathbb{A}_p$ weights and the reverse Hölder classes of weights.

Definition 1.4. *A weight w is a non-negative locally integrable function. We say that $w \in \mathbb{A}_p$, $1 < p < \infty$, if there exists a positive constant C such that for every cube Q,*

$$\Big(⨍_Q w\Big)\Big(⨍_Q w^{1-p'}\Big)^{p-1} \leq C,$$

where we have used the notation $⨍_Q w = \frac{1}{|Q|}\int_Q w$ for a cube Q.

For $p = 1$, we say that $w \in \mathbb{A}_1$ if there is a positive constant C such that for every cube Q,

$$⨍_Q w \leq Cw(y), \text{ for a.e. } y \in Q.$$

The infimum of the constants C in the above inequalities is called the $\mathbb{A}_p$ characteristic of ω and is denoted as $[\omega]_{\mathbb{A}_p}$.

We write $\mathbb{A}_\infty = \cup_{p\geq 1}\mathbb{A}_p$.

Definition 1.5. *A weight $w \in RH_t$ for $t \in (1,\infty)$, if there is a constant C such that for every cube Q,*

$$\Big(⨍_Q w^p\Big)^{1/p} \leq C\Big(⨍_Q w\Big).$$

In [2], the author proved the following weighted estimates for the multi-frequency Calderón-Zygmund operators whose kernels satisfy the regularity condition (1.1.2).

Theorem 1.6. [2] *Let Θ be a collection of N frequencies. For $p \in (2,\infty)$, $s \in [2,\infty)$ and $t \in (1,\infty)$, the multi-frequency Calderón-Zygmund operator T relatively to Θ is bounded on $L^p(w)$ for every weight $w \in RH_{t'} \cap \mathbb{A}_{\frac{p}{s}}$ with the bound*

$$\|T\|_{L^p(w)\to L^p(w)} \leq C_w N^{\frac{tp}{2s}+(\frac{1}{2}-\frac{1}{s})}.$$

We would like to emphasize that the above result is interesting only when the exponent of N is strictly smaller than one. Also, this result addresses the issue of weighted boundedness of multi-frequency operators only for a subclass of $\mathbb{A}_p$. In this paper, we not only provide a complete result about the weighted boundedness of the multi-frequency $\omega-$Calderón-Zygmund operators, but also obtain sharp bounds for the exponent of N and the $\mathbb{A}_p$ characteristic of the weights.

The proof of Theorem 1.6 uses a new sharp maximal function relatively to the collection Θ, which is an interesting object of study in its own right. We refer the interested reader to Section 3 of [2] for more details. Another key ingredient in the proof of Theorem 1.6 is the abstract good $\lambda-$inequality (Theorem 3.1 in [1]). However, it does not seem to be possible to further refine the existing proof of Theorem 1.6 to obtain better estimates for the operators under consideration. In this paper, we propose to use the modern techniques of domination by sparse operators in the context of multi-frequency Calderón-Zygmund operators. This approach turns out to be effective and yields the best possible answers to the questions raised here about the multi-frequency $\omega-$ Calderón-Zygmund operators.

2. Sparse operator domination of multi-frequency $\omega-$Calderón-Zygmund operators

The main result of this paper is the following.

Theorem 2.1. *Let $\Theta = \{\xi_1, \xi_2, ...\xi_N\}$ be a collection of N frequencies, ω be a modulus of continuity satisfying Dini condition (1.0.1) and T be a multi-frequency $\omega-$Calderón-Zygmund operator relatively to Θ. Then for $1 < p < \infty$, we have the following sharp weighted bounds for the operator T on $L^p(w)$*

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim N^{\frac{1}{2}}[w]_{A_p}^{max(1,\frac{1}{p-1})}.$$

for all $w \in \mathbb{A}_p$, with the implicit constant depending only on the dimension and the modulus of continuity ω.

Moreover, the exponents of N and $[w]_{A_p}$ in the above estimate are optimal.

In order to prove Theorem 2.1, we would require un-weighted L^p-estimates for the multi-frequency $\omega-$ Calderón-Zygmund operators with optimal control on the constants involving the parameter N. Therefore, we first prove the un-weighted L^p-boundedness properties of the operator T. In fact, in view of the Marcinkiewicz interpolation theorem, it is sufficient to prove the weak-type $(1,1)$ boundedness of the operator T with the optimal exponent of N.

Theorem 2.2. *Let $\Theta = \{\xi_1, \xi_2, ...\xi_N\}$ be a collection of N frequencies of $\mathbb{R}^n, \omega$ be a modulus of continuity satisfying Dini condition (1.0.1) and T be a multi-frequency $\omega-$Calderón-Zygmund operator relatively to the collection Θ, then T is of weak type $(1,1)$ with*

$$\|T\|_{L^1\to L^{1,\infty}} \lesssim N^{1/2}. \tag{2.0.1}$$

Proof. The multi-frequency Calderón-Zygmund decomposition Theorem 1.2 is the key tool in the proof. We closely follow the arguments from [2] in order to complete the proof of this theorem. Therefore, we would skip certain details and only provide the key steps in the proof with the necessary details.

For $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, we invoke the multi-frequency Calderón-Zygmund decomposition and write $f = g + \sum_Q b_Q$ for the disjoint collection $\Omega = (Q)$ with all the properties as listed in Theorem 1.2. The standard arguments (see [2] for details) gives us that for the distribution function $\Gamma_\lambda = \{x : |Tf(x)| > \lambda\}$ we have

$$|\Gamma_\lambda| \lesssim N^{1/2}\|f\|_{L^1}\lambda^{-1} + \lambda^{-1}\sum_Q \|T(b_Q)\|_{L^1((4Q)^c)}.$$

Let $x \in \left(\cup_{Q\in\Omega} 4Q\right)^c$. By the integral representation and the simultaneous cancellation property for all frequencies in Θ, we get

$$T(b_Q)(x_0) = \sum_{j=1}^{N}\int_{3Q} e^{i\xi_j.(x-y)}(\tilde{K}_j(x,y) - \tilde{K}_j(x,c(Q)))b_Q(y)dy,$$

where $\tilde{K}_j(x,y) = K_j(x,y)e^{-i\xi_j.(x-y)}$. By Dini-regularity condition (1.1.3) of the kernels K_j,

$$\begin{aligned}\|T(b_Q)\|_{L^1((4Q)^c)} &\lesssim \int\int_{|x-y|\geq r(Q)} \omega\left(\frac{|y-c(Q)|}{|x-y|}\right)\frac{1}{|x-y|^n}dx|b_Q(y)|dy\\ &\lesssim \|\omega\|_{Dini}\|b_Q\|_{L^1}\\ &\leq \|\omega\|_{Dini}|Q|\lambda.\end{aligned}$$

Thus $|\Gamma_\lambda| \lesssim (c_n + \|\omega\|_{Dini})\lambda^{-1}N^{1/2}\|f\|_{L^1}$. This completes the proof estimate (2.0.1). □

Note that the Marcinkiewicz interpolation theorem applied to the weak-type $(1,1)$ and strong type $(2,2)$ estimates for the operator T gives us that $\|T\|_{L^p\to L^p} \lesssim N^{\frac{1}{p}-\frac{1}{2}}$ for all $1 < p < 2$ and then by duality we obtain the desired estimate for $2 < p < \infty$, the remaining set of exponents. Moreover, the following proposition suggests that the exponent of N in the above theorem is optimal.

Proposition 2.3. [8]*: For each $N \in \mathbb{N}$ and $p \in (1,2)$ there is a choice of signs $(\epsilon_n)_{1\leq n\leq N}$ such that if $\widehat{f_N} = 1_{[0,N]}$, then*

$$\left\|\int \widehat{f_N}(\xi)\sum_{n=0}^{N-1}\epsilon_n 1_{[n,n+1]}(\xi)e^{2\pi i\xi x}d\xi\right\|_{L^p(\mathbb{R})} \geq CN^{\frac{1}{p}-\frac{1}{2}}\|f_N\|_{L^p(\mathbb{R})}.$$

2.1. **Domination by sparse operators.** We say that a family $\mathcal{S}$ of cubes from $\mathbb{R}^n$ is η-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$ and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. Then for a non-negative locally intergrable function f in $\mathbb{R}^n$, the sparse operator is defined by

$$\mathcal{A}_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q f\right)\chi_Q(x).$$

The sparse operator satisfies the following sharp weighted inequalities on $L^p(w)$ for all $w \in \mathbb{A}_p$.

Theorem 2.4. [14] *For every η-sparse family $\mathcal{S}$ and for all $1 < p < \infty$,*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)} \leq c_{n,p,\eta} [w]_{\mathbb{A}_p}^{max(1,\frac{1}{p-1})} \|f\|_{L^p(w)}.$$

Therefore, in order to prove Theorem 2.1, it suffices to obtain almost everywhere pointwise domination of $T(f)(x)$ by a sparse operator. We shall follow the techniques from [10], [11] and [14] in order to obtain the pointwise a.e. domination of the operators under consideration by suitable sparse operators.

Similar to their argument we prove for the multi-frequency ω-Calderón-Zygmund operator and the proof is based on the pointwise bounds for a grand maximal truncated operator $\mathcal{M}_T$ given by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

where the supremum is being taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

For a given cube Q_0 and $x \in Q_0$, the local version of $\mathcal{M}_T$ is defined by

$$\mathcal{M}_{T,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \operatorname*{ess\,sup}_{\xi \in Q} |T(f\chi_{3Q_0 \setminus 3Q})(\xi)|.$$

We now prove the pointwise bounds of the grand maximal truncated operator $\mathcal{M}_T$ and thereby the bounds of $\mathcal{M}_{T,Q_0}$- the local version of $\mathcal{M}_T$.

Lemma 2.5. *Let $\Theta = \{\xi_1, \xi_2, ...\xi_N\}$ be a collection of N frequencies, ω be a modulus of continuity satisfying Dini condition (1.0.1) and T be a multi-frequency $\omega-$Calderón-Zygmund operator relatively to Θ. Then*

$$\mathcal{M}_T f(x) \leq C_{n,r}(\|\omega\|_{Dini} + N^{\frac{1}{2}}) Mf(x) + (M(|Tf|^r)(x))^{1/r} \tag{2.1.2}$$

for $0 < r < 1$. Here Mf denotes the classical Hardy-Littlewood maximal function.

Proof. Let $x \in Int\ Q$ and $\xi \in Q$. Denote $B(x)$ the closed ball centered at x of radius 2diam Q. Observe that $3Q \subset B(x)$ and

$$|T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \leq |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi) - T(f\chi_{\mathbb{R}^n \setminus 3Q})(z)| + |T(f\chi_{\mathbb{R}^n \setminus 3Q})(z)|. \tag{2.1.3}$$

For any $z \in B_{diam\ Q}(x)$, we have

$$\begin{aligned}
&|T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi) - T(f\chi_{\mathbb{R}^n \setminus 3Q})(z)| \\
&\leq |\int \sum (K_j(\xi, y) - K_j(z, y))(f\chi_{\mathbb{R}^n \setminus B(x)})(y)dy| \\
&\quad + |\int \sum (K_j(\xi, y) - K_j(z, y))(f\chi_{B(x) \setminus 3Q})(y)dy| \\
&\leq \int_{\mathbb{R}^n \setminus B(x)} \Big(|\sum (K_j(\xi, y) - K_j(x, y))| + |\sum (K_j(z, y) - K_j(x, y))| \Big) |f(y)|dy \\
&\quad + \int_{B(x) \setminus 3Q} \Big(|\sum (K_j(\xi, y) - K_j(x, y))| + |\sum (K_j(z, y) - K_j(x, y))| \Big) |f(y)|dy
\end{aligned}$$

Note that for $\xi \in Q \subset B(x)$, $|\xi - x| \leq$diam Q and since $z \in B_{diam\ Q}(x)$, $|z - x| \leq$diam Q. Therefore, by the Dini- regularity condition of kernel and the fact

that ω is increasing, we get

$$
\begin{aligned}
&|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)-T(f\chi_{\mathbb{R}^n\setminus 3Q})(z)|\\
&\leq \int_{|x-y|>2diam\ Q}|f(y)|2\omega\Big(\frac{2diam\ Q}{|x-y|}\Big)\frac{1}{|x-y|^n}dy+\frac{2c\omega(1)}{(diam\ Q)^n}\int_{B(x)}|f|\\
&\leq \sum_{k=1}^{\infty}\Big(\frac{C_n}{(2^k diam\ Q)^n}\int_{2^kB(x)}|f|\Big)\omega(2^{-k})+\frac{C_n\omega(1)}{(diam\ Q)^n}\int_{B(x)}|f|\\
&\leq C_n\|\omega\|_{Dini}Mf(x).
\end{aligned}
$$

Hence, the inequality (2.1.3) takes the form

$$
\begin{aligned}
|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &\leq C_n\|\omega\|_{Dini}Mf(x)+|T(f\chi_{\mathbb{R}^n\setminus 3Q})(z)|\\
&= C_n\|\omega\|_{Dini}Mf(x)+\Big|\int K(z,y)f(y)dy-\int_{3Q}K(z,y)f(y)dy\Big|\\
&\leq C_n\|\omega\|_{Dini}Mf(x)+|T(f)(z)|+|T(f\chi_{3Q})(z)|,
\end{aligned}
$$

where $K(x,y)=\sum_j K_j(x,y)$. This gives us that for $0<r<1$,

$$|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|^r\leq C_n^r\|\omega\|_{Dini}^r Mf(x)^r+|T(f)(z)|^r+|T(f\chi_{3Q})(z)|^r.$$

Integrating the above with respect to $z\in B_{diamQ}(x)$, then dividing by $|B_{diamQ}(x)|$ and rising to the power $\frac{1}{r}$, we obtain

$$
\begin{aligned}
|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &\leq 3^{1/r}\Bigg[C_n\|\omega\|_{Dini}Mf(x)+\Big(\frac{1}{|B_{diamQ}(x)|}\int_{B_{diamQ}(x)}|T(f)(z)|^r dz\Big)^{\frac{1}{r}}\\
&\quad+\Big(\frac{1}{|B_{diamQ}(x)|}\int_{B_{diamQ}(x)}|T(f\chi_{3Q})(z)|^r dz\Big)^{\frac{1}{r}}\Bigg]\\
&\leq 3^{1/r}\Bigg[C_n\|\omega\|_{Dini}Mf(x)+M(|T(f)|^r)(x)^{\frac{1}{r}}\\
&\quad+\frac{\|T\|_{L^1\to L^{1,\infty}}}{(1-r)^{1/r}|B_{diamQ}(x)|}\|f\chi_{3Q}\|_{L^1}\Bigg].
\end{aligned}
$$

Here for the third term on the right in the above we have used the estimate

$$\int_{B_{diamQ}(x)}|T(f\chi_{3Q})(z)|^r dz\leq\frac{\|T\|_{L^1\to L^{1,\infty}}^r|B_{diamQ}(x)|^{1-r}}{(1-r)}\|f\chi_{3Q}\|_{L^1}^r.$$

Thus we obtain

$$|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|\leq 3^{1/r}\Bigg[\Big(C_n\|\omega\|_{Dini}+\frac{\|T\|_{L^1\to L^{1,\infty}}}{(1-r)^{1/r}}\Big)Mf(x)+M(|T(f)|^r)(x)^{\frac{1}{r}}\Bigg].$$

Substituting this estimate into (2.0.1), we complete the proof of Lemma 2.5. $\square$

Remark 2.6. *Note that*
$\|M(|Tf|^r)^{1/r}\|_{L^{1,\infty}}\leq C_{n,r}\|T\|_{L^1\to L^{1,\infty}}\|f\|_{L^1}$ *and* $\|M(|Tf|^{1/2})^2\|_{L^p}\leq C_n max(p,\frac{1}{1-p})\|T\|_{L^p}\|f\|_{L^p}$.

We are finally in a position to show the pointwise almost everywhere domination of the multi-frequency ω-Calderón-Zygmund operator by a sparse operator.

Theorem 2.7. *Let T be a ω-Calderón-Zygmund operator relative to the frequency set $\Theta = (\xi_1, ...\xi_N)$ with ω satisfying the Dini condition (1.0.1). Then for every compactly supported $f \in L^1(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$|Tf(x)| \lesssim N^{\frac{1}{2}} \mathcal{A}_{\mathcal{S}}|f|(x), \tag{2.1.4}$$

Proof. We claim that for a.e. $x \in Q_0$,

$$|T(f\chi_{3Q_0})(x)| \leq c_n N^{\frac{1}{2}}|f(x)| + \mathcal{M}_{T,Q_0}f(x). \tag{2.1.5}$$

Suppose that $x \in$int Q_0, and let x be a point of approximate continuity of $T(f\chi_{3Q_0})$. Then for every $\epsilon > 0$, the sets

$$E_s(x) = \{y \in B_s(x) : |T(f\chi_{3Q_0})(y) - T(f\chi_{3Q_0})(x)| < \epsilon\}$$

satisfy $\lim_{s\to 0} \frac{|E_s(x)|}{|B_s(x)|} = 1$, where $B_s(x)$ is the open ball centered at x of radius s.

Let $Q_s(x)$ denote the smallest cube centered at x and containing the ball $B_s(x)$. Note that we may take $s > 0$ small so that $Q_s(x) \subset Q_0$. Then for a.e. $y \in E_s(x)$,

$$\begin{aligned} |T(f\chi_{3Q_0})(x)| &\leq |T(f\chi_{3Q_0})(y)| + |T(f\chi_{3Q_0})(x) - T(f\chi_{3Q_0})(y)| \\ &\leq |T(f\chi_{3Q_s(x)})(y)| + |T(f\chi_{3Q_0(x)\setminus 3Q_s(x)})(y)| + \epsilon \\ &\leq \operatorname*{ess\,inf}_{y\in E_s(x)} |T(f\chi_{3Q_s(x)})(y)| + \mathcal{M}_{T,Q_0}f(x) + \epsilon. \end{aligned}$$

Therefore, applying the weak type $(1,1)$ boundedness of T yields: For $\alpha = \operatorname*{ess\,inf}_{y\in E_s(x)} |T(f\chi_{3Q_s(x)})(y)|$,

$$\begin{aligned} |E_s(x)| &= |\{y \in E_s(x) : |T(f\chi_{3Q_s(x)})(y)| > \alpha\}| \\ &\leq \|T\|_{L^1\to L^{1,\infty}} \alpha^{-1} \int |f|\chi_{3Q_s(x)})(y). \end{aligned} \tag{2.1.6}$$

Thus

$$|T(f\chi_{3Q_0})(x)| \leq \|T\|_{L^1\to L^{1,\infty}} \frac{1}{|E_s(x)|} \int_{3Q_s(x)} |f| + \mathcal{M}_{T,Q_0}f(x) + \epsilon.$$

Assuming additionally that x is a Lebesgue point of f and letting subsequently $s \to 0$ and $\epsilon \to 0$, we obtain

$$|T(f\chi_{3Q_0})(x)| \leq c_n \|T\|_{L^1\to L^{1,\infty}} |f(x)| + \mathcal{M}_{T,Q_0}f(x).$$

By substituting the weak-type $(1,1)$ bound for the operator norm from (2.0.1), we get (2.1.5).

We next prove that for a fixed cube $Q_0 \subset \mathbb{R}^n$, there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,

$$|T(f\chi_{3Q_0})(x)| \leq c_n C_T \sum_{Q\in\mathcal{F}} |f|_{3Q}\chi_Q(x), \tag{2.1.7}$$

where $C_T = N^{\frac{1}{2}} + \|\omega\|_{Dini}$ and $\mathcal{D}(Q_0)$ is the set of all dyadic cubes with respect to Q_0, that is, the cubes obtained by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes.

As in the proof of Theorem 3.1 in [14], it is enough to prove that there exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and

$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j} \text{ a.e. } x \in Q_0$$

Indeed, iterating the above estimate, we obtain the claim with $\mathcal{F} = \{P_j^k\},\ k \in \mathbb{Z}_+$ where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ and $\{P_j^k\}$ are the cubes obtained at the k^{th} stage of the iterative process.

Since

$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} = |T(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j} + |T(f\chi_{3Q_0})(x)|\chi_{\cup_j P_j},$$

it is enough to prove that

$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j} \leq c_n C_T |f|_{3Q_0}. \tag{2.1.8}$$

Let c_n be a constant to be chosen at a later time. Consider the set

$$E = \{x \in Q_0 : |f(x)| > c_n|f|_{3Q_0}\} \cup \{x \in Q_0 : \mathcal{M}_{T,Q_0}f(x) > c_n C_T|f|_{3Q_0}\}.$$

By the choice of C_T and the weak-type $(1,1)$ bound $\|\mathcal{M}_T\|_{L^1\to L^{1,\infty}} \leq \alpha_n C_T$, one can choose c_n in such a way that the set E satisfies the estimate

$$|E| \leq \frac{1}{2^{n+2}}|Q_0|.$$

For, we have

$$|\{x \in Q_0 : \mathcal{M}_{T,Q_0}f(x) > c_n C_T|f|_{3Q_0}\}| \leq \frac{\alpha_n C_T}{c_n C_T|f|_{3Q_0}}\int_{Q_0}|f| \leq \frac{\alpha_n|3Q_0|}{c_n}.$$

Similarly,

$$|\{x \in Q_0 : |f(x)| > c_n|f|_{3Q_0}\}| \leq \frac{1}{c_n|f|_{3Q_0}}\int_{Q_0}|f| \leq \frac{|3Q_0|}{c_n}.$$

Therefore, a suitable choice of c_n would guarantee the above estimate on $|E|$.

Now, the classical Calderón-Zygmund decomposition applied to the function χ_E with respect to the cube Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and $\chi_E(x) \leq \frac{1}{2}$ for all x in the complement of $\cup_j P_j$, that is, $|E\setminus \cup_j P_j| = 0$. It follows that

$$\sum_j |P_j| = 2^{n+1}\sum |E \cap P_j| \leq 2^{n+1}|E| \leq \frac{1}{2}|Q_0|$$

and $P_j \cap E^c \neq \emptyset$ as $|P_j \cap E| \leq \frac{1}{2}|P_j|$. Therefore,

$$\operatorname*{ess\,sup}_{\xi\in P_j}|T(f_{3Q_0\setminus 3P_j})(\xi)| \leq c_n C_T|f|_{3Q_0}. \tag{2.1.9}$$

Further, by the choice of C_T and the estimate (2.1.5), we have

$$|T(f\chi_{3Q_0})(x)| \leq c_n C_T|f|_{3Q_0} \text{ a.e. } x \in Q_0\setminus \cup_j P_j.$$

Since the cubes P_j's are disjoint, the above equation together with the estimate (2.1.9) yields the required inequality (2.1.8).

Having obtained the pointwise domination locally, we need to extend it to whole of $\mathbb{R}^n$. This is achieved by following the standard arguments as in [14]. We consider a partition of $\mathbb{R}^n$ by cubes R_j such that supp $(f) \subset 3R_j$ for each j. Such a partition may be obtained in the following way.

Take a cube Q_0 such that supp $(f) \subset Q_0$ and note that $3Q_0\setminus Q_0$ is covered by $3^n - 1$ congruent cubes R_j. For each cube R_j, we have $Q_0 \subset 3R_j$. Next, in the

same way cover $9Q_0 \backslash 3Q_0$ and so on. The union of resulting cubes including Q_0 will satisfy the desired property.

After having such a partition we apply (2.1.7) to each cube R_j of the collection to obtain a $\frac{1}{2}-$sparse family $\mathcal{F}_j \subset \mathcal{D}(R_j)$ such that (2.1.7) holds with Tf in the left hand side for a.e. $x \in R_j$. Hence setting $\mathcal{F} = \cup_j \mathcal{F}_j$, we obtain

$$|Tf(x)| \lesssim C_T \sum_{Q \in \mathcal{F}} |f|_{3Q} \chi_Q(x).$$

Finally, as in [14] it is easy to verify that the collection $\mathcal{S} = \{3Q : Q \in \mathcal{F}\}$ is η-sparse with $\eta = \frac{1}{2.3^n}$. This completes the proof of the pointwise estimate (2.1.4). □

Remark 2.8. *(Sharpness of constants in Theorem 2.1) The $\mathbb{A}_p$ characteristic constant $[w]_{\mathbb{A}_p}^{max(1,\frac{1}{p-1})}$, being sharp in weighted inequalities for a single Calderón-Zygmund operator, implies the same for the multi-frequency $\omega-$Calderón-Zygmund operator. Next, we see that the exponent $1/2$ of N is also optimal in Theorem 2.1. For, if we could get a smaller exponent, say $N^{\frac{1}{2}-\epsilon}$, for the operator norm $\|T\|_{L^p_w \to L^p_w}$ for some $1 < p < \infty$, then the sharp form of the extrapolation theorem implies that the operator T is bounded on L^p with $\|T\|_{L^p \to L^p} \lesssim N^{\frac{1}{2}-\epsilon}$ for all $1 < p < \infty$. This estimate together with the Proposition 2.3 would then lead to a contradiction by choosing p close enough to 1. Hence, the exponent of N in Theorem 2.1 cannot be taken to be smaller than $1/2$.*

References

[1] Auscher, P., and Martell, J.M. "Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights." *Adv. Math. 212 no. 1225–276.* (2007)

[2] Bernicot, F. "Multi-frequency Calderón-Zygmund analysis and connexion to Bochner-Riesz multipliers" *Harmonic analysis and partial differential equations, Contemp. Math., Amer. Math. Soc., 612, 29-43.* (2014)

[3] Beznosova,O. V. Linear bound for the dyadic paraproduct on weighted Lebesgue space $L^2(w)$. *Journal of Functional Analysis, 255, no. 4, 994-1007.*(2008)

[4] Borwein, P., and Erdelyi, T. "Nikolskii-type inequalities for shift invariant function spaces" *Proc. Amer. Math. Soc. 134, no. 11, 3243–3246.* (2006)

[5] Conde-Alonso, J. M., and Rey, G. M. "A pointwise estimate for positive dyadic shifts and some applications" *Math. Ann., 365, no. 3-4, 1111-1135.* (2016)

[6] Cruz-Uribe, D., Martell, J. M., and Perez, C. "Sharp weighted estimates for approximating dyadic operators" *Electron. Res. Announc. Math. Sci., 17, 12-19.* (2010)

[7] Cruz-Uribe, D., Martell, J. M., and Perez, C. Sharp weighted estimates for classical operators. *Advances in Mathematics 229, no. 1, 40841.*(2012)

[8] Demeter, C. "On some maximal multipliers in L^p" *Rev. Mat. Iberoamericana 26, 3, 947-964.* (2010)

[9] Hytonen, P. T. "The sharp weighted bound for general Calderón-Zygmund operators" *Ann. of Math. (2) 175 (3), 1473-1505.* (2012)

[10] Hytonen, P. T., Roncal, L., and Tapiola, O. "Quantitative weighted estimates for rough homogeneous singular integrals" *Preprint. arXiv:1510.05789*. (2015)

[11] Lacey, M. T., "An elementary proof of the A_2 bound" *Israel J. Math., to appear.* (2015)

[12] Lacey, M. T., Petermichl, S., and Reguera, M. C. "Sharp A_2 inequality for Haar shift operators" *Math. Ann. 348 (1), 127-141.* (2010)

[13] Lerner, A. K. "A simple proof of the A_2 conjecture" *Int. Math. Res. Not. (IMRN) 14, 3159-3170.* (2013)

[14] Lerner, A. K. "On pointwise estimates involving sparse operators" *New York J. Math. 22, 341-349.* (2016)

[15] Lerner, A. K., and Nazarov, F. "Intuitive dyadic calculus: the basics" *Preprint. arXiv:1508.05639* (2015)

[16] Nazarov, F., Oberlin, R., and Thiele, C. "A Caldrón Zygmund decomposition for multiple frequencies and an application to an extension of a lemma of Bourgain" *Math. Res. Lett. 17, 3, 529-545.* (2010)

[17] Oberlin R., and Thiele, C., "New uniform bounds for a Walsh model of the bilinear Hilbert transform" *Indiana Univ. Math. J. 60, no. 5, 1693–1712.* (2011)

[18] Petermichl, S. "The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p-characteristic." *Amer. J. Math. 129, no. 5, 1355–1375.* (2007)

[19] Petermichl, S., "The sharp weighted bound for the Riesz transform." *Proc. Amer. Math. Soc. 136, no. 4, 1237–1249.* (2008)

Department of Mathematics, Indian Institute of Science Education and Research, Bhopal-462 066

E-mail address: saurabhk@iiserb.ac.in, kssenthilraani@gmail.com